



# SQUEEZED STATES IN BLACK-HOLE EVAPORATION BY ANALYTIC CONTINUATION

Andrew N. S. J. Farley

DAMTP, Centre for Mathematical Sciences, University of Cambridge,  
Wilberforce Road, Cambridge CB3 0WA, UK.

A.N.S.J.Farley@damtp.cam.ac.uk

February 7, 2008

## Abstract

We compute the semi-classical quantum amplitude to go from an initial spherically symmetric bosonic matter and gravitational field configuration to a final radiation configuration, corresponding to the relic Hawking radiation from a non-rotating, chargeless black hole which evaporates completely. This is obtained via the classical action integral which is solely a boundary term. On discretising the classical action, the quantum amplitude can be expressed in terms of generalised coherent states of the harmonic oscillator. A squeezed-state representation is obtained by complexifying the proper time separation  $T$  at spatial infinity between the initial and final space-like hypersurfaces. Such a procedure is deemed necessary as the two-surface problem for Dirichlet boundary data and wave-like perturbations is not well posed. We find that infinitesimal rotation into the lower complex  $T$  plane is equivalent to a highly-squeezed final state for the relic radiation, similar to the relic gravitational-wave background in cosmology. This final state is a pure state, and so the unpredictability associated with the final momentarily-naked singularity is avoided. The cosmological analogy is the tunnelling from an initial smooth Euclidean or timeless state to a classical universe. The high-squeezing limit corresponds to a final state of the Hawking flux which is indistinguishable from a stochastic collection of standing waves. The phases conjugate to the field amplitudes are squeezed to discrete values. We also discuss the entropy of the final radiation in the high-squeezing limit.

# 1 Introduction

Much has been achieved in the application of Feynman's quantum-amplitude formalism to black-hole evaporation. In a semi-classical approximation, Hartle and Hawking [1] pioneered this technique, proving, by analytic continuation in the complexified space-time, that the amplitude for a black hole to emit a particle was proportional to the amplitude for the black hole to absorb a particle. The proportionality factor is just a Boltzmann factor, whence one determines that a black hole of initial mass  $M_I$  can come into equilibrium with the exterior radiation only at the Hawking temperature  $T_H = \hbar c^3 / 8\pi G M_I k_B$ . Gibbons and Hawking [2] again employed semi-classical path-integral methods to re-derive Hawking radiance and the Bekenstein-Hawking entropy formula.

In this paper, we determine the semi-classical wave functional of specified boundary data in the context of classical-like coherent states and purely quantum-mechanical squeezed states. The emphasis will be on the final radiation which remains after a black hole has evaporated completely, and its comparisons with the relic Cosmic Microwave Background Radiation (CMBR) induced by cosmological perturbations.

It is well-known that particle creation by black holes has many similarities with cosmological particle creation, despite the lack of asymptotic flatness in the cosmological scenario. Cosmological and black-hole particle creation both require a time dependence in the metric. The time-dependent background gravitational field (also known as the 'pump' field) is a result of the non-linearity of Einstein's equations, which is displayed through the coupling of the gravitational wave field, say, with the evolving background space-time. One is also aware of cosmological No-Hair Theorems, analogous to the black-hole case, as well the presence of singularities, at some finite time in the past in the cosmological case, and hidden behind event horizons in the black-hole case.

Rather than employ a time-dependent background metric, black-hole evaporation can be interpreted as a tunnelling process, whereby the strong gravitational field in the neighbourhood of the future event horizon promotes vacuum fluctuations into real particles. Positive-energy particles are detected by observers at infinity, while negative-energy particles tunnel through the event horizon reducing the mass of the black hole. Eventually the black hole presumably disappears completely (we assume in this work that there are no black-hole relics) leaving only radiation and an almost flat space-time. The singularity inside the black hole is momentarily naked prior to total disappearance [20], and is deemed to take with it the information about possible collapse configurations which created the black hole. This is the information-loss paradox.

In cosmology, near the start of inflation, the quantum vacuum state of each particle with oppositely directed momenta and short wavelengths is the adiabatic ground state. This is related to the assumption that the universe was in a maximally-symmetric state at some time in the past which does not contain a curvature singularity [3]. Due to the accelerated expansion of the universe during inflation, quantum fluctuations are amplified into macroscopical or classical perturbations. The early-time fluctuations lead to the formation of large-scale structure in the universe, and also contribute to the temperature anisotropies in the CMBR. For

modes whose wavelength is much greater than the Hubble radius, the final state for the perturbations is a highly-correlated two-mode squeezed state, with pairs of field quanta produced at late times with opposite momenta [4]. Tensor fluctuations in the metric, for example, are predicted to give rise to relic gravitational waves. Electromagnetic waves cannot be squeezed during cosmological expansion in the same way the tensor (metric) perturbations are, because they do not interact with the external gravitational field in the same way as gravitational waves do.

For the non-rotating, chargeless black hole we shall be considering, the only parameter characterising the black hole is its initial mass  $M_I$ . In the adiabatic approximation for evaporating black holes, for much of the evaporation, frequencies typically exceed  $\frac{|\dot{m}|}{m}$ , the inverse time-variation scale for the black-hole mass. That is, the wave period is much smaller than the timescale of variations of the background gravitational field. The black hole interacts negligibly with the emitted particles, and the time between successive emissions is comparable with the black hole mass [5]. Prior to the final disappearance of the hole, however, particle frequencies are of the order  $m$ , and the frequency of variations of the background space-time is comparable to particle frequencies. The particles then interact strongly with the evanescent black hole.

In the cosmological case, a natural length scale is the time-dependent inverse Hubble parameter  $H^{-1} = \frac{a}{\dot{a}}$  ( $a(t)$  is the Friedmann-Robertson-Walker scale factor), with the adiabatic approximation  $k \gg H$ , or such that the wavelengths of the perturbations are less than the Hubble radius  $aH^{-1}$ . When the wavelength is comparable with or longer than the Hubble radius, the amplification of the zero-point fluctuations takes place. In addition, both the ADM mass and Hubble parameter control the redshifting of the radiation in the background space-time.

Due to the Schrödinger evolution, cosmological perturbations (rotational, density and gravitational) in an initial vacuum state are transformed into a highly-squeezed vacuum state with many particles having a large variance in their amplitude (particle number) and small (squeezed) phase variations. At small wavelengths, the squeezing of cosmological perturbations may be suppressed, while it should be present at long wavelengths, particularly for gravitational waves [38]. These perturbations also induce the large-angular-scale anisotropies observed in the CMBR. Their wavelengths today are of the order of or greater than the Hubble radius. The amplification of the initial zero-point fluctuations gives rise to standing waves with a fixed phase, rather than travelling waves. The relic perturbations in the high-squeezing or WKB limit can be described as a stochastic collection of standing waves. However, it has been suggested that the squeezed-state formalism engenders no new physics [24].

A prominent theme of this paper is the squeezed-state formalism applied to black-hole evaporation, with reference to the comparable inflationary-cosmology scenario. Grishchuk and Sidorov [4] first achieved a squeezed-state representation for Hawking radiation. Their approach, however, referred to final quantum states describing the particles escaping to infinity and those falling through the event horizon. In this paper, the squeezed-state approach is related to the radiation which remains after a black hole has evaporated completely. In an adiabatic approximation, the fixed phases correspond to discrete frequencies in the remnant Hawking radiation from

the evanescent black hole. This high-squeezing feature, we argue, can potentially be observed. The standing waves of the highly-squeezed final state of the Hawking radiation originates from choosing to set Dirichlet boundary data on initial and final space-like hypersurfaces. Such a problem is not well posed, and so we propose complexifying the proper time interval at spatial infinity in order to obtain a well-posed problem. This, we believe, has the effect of avoiding the problems associated with the final naked singularity which may be present prior to the final disappearance of the black hole. Thus, we conclude that a highly-squeezed final state, which is a pure state, is related to the avoidance of the naked singularity, and its associated unpredictability.

After a brief summary of our two-surface method in Section 2, in Section 3 we write the quantum amplitude for linearised spin-0, 1, 2 fields in a unified form. Coherent states and generalised coherent states are related to the quantum amplitude in Section 4. Section 5 introduces the squeezed-state formalism into the theory *via* a complexification technique. In Section 6, we discuss issues of entropy and classical behaviour. Throughout we employ Planckian units  $G = c = \hbar = 1$ .

## 2 Two-Surface Formulation

This section is a summary of results which will be elaborated on in ref. [6].

Our approach to black-hole evaporation is through a two-surface boundary-value problem. We consider the quantum-mechanical decay of a chargeless, non-rotating black hole into almost flat space-time and purely outgoing radiation. Data for spin-0,  $\frac{1}{2}$ , 1,  $\frac{3}{2}$ , 2 perturbations is set on an initial space-like hypersurface  $\Sigma_I$ , just prior to the collapse of an initially diffuse (spherical) ‘star’ to a black hole, and a final space-like hypersurface  $\Sigma_F$ , sufficiently far to the future of  $\Sigma_I$  that it catches the relic Hawking radiation. The black hole evaporates completely into predominantly massless particles, with the final total (ADM) energy equal to the initial mass  $M_I$ .

The classical action  $S^{(cl)}$ , i.e., the action evaluated at the solution of the field equations, reduces to solely a boundary term, where the boundary comprises  $\Sigma_I$ ,  $\Sigma_F$  and a time-like boundary  $\Sigma^\infty$  ( $\{r = R_\infty\}$ ) located at large radius connecting  $\Sigma_I$  to  $\Sigma_F$ , the proper time separation at spatial infinity being denoted by  $T$ . For simplicity, we considered initial data of very weak perturbations, i.e., such that the matter and gravitational fields were initially spherically symmetric. The final data comprises non-zero perturbations corresponding to the relic radiation after the black hole has disappeared.

Let us specialise to the integer-spin perturbations; the fermions will be considered in a future paper. For the wave-like field equations satisfied by the perturbations, the ‘Dirichlet’ boundary-value problem on a purely Lorentzian background space-time is not well-posed [7]. That is, there may be no solution to the field equations with this boundary data, or an infinite number of solutions. Adopting a ‘Euclidean’ strategy, we rotate  $T$  into the lower complex plane. Consequently, one might then expect there to be a unique classical solution to the nearly-Lorentzian field equations. This is just Feynman’s  $\pm i\varepsilon$  prescription in quantum field theory.

In a neighbourhood of  $\Sigma_I$  and  $\Sigma_F$ , one may employ an adiabatic approximation, whereby most of the radiation frequencies exceed the rate of change of the

background space-time. One may then ‘Fourier’ decompose the perturbations into standing waves, so that the boundary conditions on  $\Sigma_I$  and  $\Sigma_F$  are satisfied. For example, the real linearised massless scalar field  $A^{(1)}(x)$  which vanishes on  $\Sigma_I$  and is non-zero on  $\Sigma_F$  is written as

$$A^{(1)}(x) = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^l \int_{-\infty}^{\infty} dk a_{0klm+} \xi_{0kl+}(t, r) \frac{\sin(kt)}{\sin(kT)} Y_{lm}(\Omega). \quad (1)$$

The background line-element is spherically-symmetric on a time average and has the time-dependent Schwarzschild-like form

$$ds^2 = -e^{b(t,r)} dt^2 + e^{a(t,r)} dr^2 + r^2 d\Omega^2, \quad (2)$$

where  $d\Omega^2$  is the two-sphere line-element. In the limit of wavelengths small compared to the background radius of curvature, the background geometry at late times is approximately a Vaidya space-time. The  $\{Y_{lm}(\Omega)\}$  are spherical harmonics with angular momentum  $l$  and azimuthal angular momentum  $m$ . The  $\{\xi_{0kl+}(t, r)\}$  are ‘radial’ functions which are functions of  $r$  and slowly-varying functions of  $t$  in an adiabatic approximation. The  $\{a_{0klm+}\}$  are a set of time-independent amplitudes (see below). Only one set of amplitudes is associated with the final state of the radiation as we suppose that there is no event horizon since the black hole disappears completely. A decomposition similar to eqn.(1) for the spin-1 and spin-2 perturbations is also possible – see ref. [6]. The boundary-value problem for the fermionic fields differs from the bosonic case due to the first-order nature of their actions.

By taking the limit  $R_{\infty} \rightarrow \infty$ , one can normalise the radial functions on  $\Sigma_F$ , assuming regularity at the origin  $\{r = 0\}$ , which implies that the radial functions are real, and a linear combination of plane waves at infinity. The classical action can then be written as a single integral over frequency. For the spin-0, 1, 2 perturbations, the classical action has an infinite number of discrete simple poles along the real frequency axis at the frequencies  $\omega_n = \frac{n\pi}{|T|}$ ,  $n = 1, 2, \dots$ . This is a manifestation of the ill-posedness of the two-surface boundary-value problem. Through our ‘Euclidean’ strategy, we can avoid the poles along the real frequency axis and subsequently obtain a probabilistic interpretation for the final boundary data. This is because the classical action acquires real and imaginary parts in the complexification procedure. At the end of our calculations, we take the limit that  $T$  approaches the real axis, a procedure intimately connected with the squeezed-state formalism. In the next section, we will compute the quantum amplitude for the integer-spin fields as a stepping stone to computing the probability density.

Additional papers [6] matched our two-surface approach with the method of Bogoliubov coefficients tailored to describing the radiation incoming from and outgoing to the null hypersurfaces  $\mathcal{I}^-$  and  $\mathcal{I}^+$  respectively.

### 3 The Quantum Amplitude

For the massless spin  $s = 0, 1, 2$  perturbations, we find that the quantum amplitude or wave functional  $\Psi[\{a_{sklmP}\}; T]$  is given by [6]

$$\Psi[\{a_{sklmP}\}; T] = N \exp \left( iS^{(cl)}[\{a_{sklmP}\}; T] \right), \quad (3)$$

where  $\{a_{sklmP}\}$  denotes a set of ‘Fourier’-like amplitudes for perturbations with spin  $s$ , frequency  $k$ , angular momentum  $l$ , azimuthal angular momentum  $m$  and polarisation  $P$ , and  $S^{(cl)}$  is the linearised total classical action. (For a massless scalar field, the polarisation can refer to the real and imaginary parts of a single complex massless scalar field.) Further,  $N$  is a  $T$ -dependent prefactor given below. We emphasise that eqn.(3) is just a semi-classical approximation to the full quantum amplitude, which is given by a path integral over field configurations which match the specified boundary data. For the simplest initial Dirichlet boundary condition that the initial perturbations are very weak, we found that

$$\begin{aligned}
S^{(cl)}[\{a_{sklmP}\}; T] &= \frac{1}{4\pi} \sum_s \sum_{l=s}^{\infty} \sum_{m=-l}^l \sum_{P=\pm} w_s \frac{(l-s)!}{(l+s)!} P \int_0^{R_\infty} dr e^{\frac{1}{2}(a-b)} \xi_{slmP} \partial_t \xi_{slmP}^* |_{\Sigma_F} \\
&\quad - \frac{1}{2} M_I T \\
&= \sum_{slmP} (-1)^s w_s \frac{(l-s)!}{(l+s)!} \int_0^\infty dk k |z_{sklP}|^2 |a_{sklmP} + (-1)^s P a_{s,-kl,-mP}|^2 \cot(kT) \\
&\quad - \frac{1}{2} M_I T.
\end{aligned} \tag{4}$$

The final term in eqn.(4) comes from the time-like boundary  $\Sigma^\infty$  and gives a plane-wave representation for black holes of fixed ADM mass  $M_I$  in the absence of perturbations. In eqn.(4),  $w_0 = 2\pi$ ,  $w_1 = \frac{1}{4}$ ,  $w_2 = \frac{1}{8}$ , and  $a_{sklmP} = P(-1)^m a_{s,-kl,-mP}^*$ .

The functions  $\{\xi_{slmP}(t, r)\}$  satisfy in an adiabatic approximation  $k \gg \frac{1}{2}|\dot{a} - \dot{b}|$  [6]

$$e^{\frac{1}{2}(b-a)} \partial_r (e^{\frac{1}{2}(b-a)} \partial_r \xi_{slmP}) - \partial_t^2 \xi_{slmP} + V_{slP}(t, r) \xi_{slmP} = 0, \tag{5}$$

where  $V_{slP}(t, r)$  is a spin-dependent potential:

$$V_{slP}(t, r) = \frac{e^{b(t,r)}}{r^2} \left[ l(l+1) + (1-s^2) \frac{2m(t, r)}{r} \right], \tag{6}$$

for  $s = 0, 1$  and odd-parity spin-2 perturbations, and a more complicated term for the even-parity metric perturbations which will not be needed in this paper. The effect of the back-reaction on the metric in the adiabatic approximation is just to replace the Schwarzschild constant mass with the time- and radially-dependent mass  $m(t, r)$  in  $V_{slP}$  where  $e^{-a(t,r)} = 1 - \frac{2m(t,r)}{r}$ . The spin-0 and spin-2 problems are very similar. In cosmology, primordial density perturbations and the relic gravitational waves are also described by similar equations. The coefficients  $\{z_{sklP}\}$  relate to the boundary conditions at spatial infinity and the regularity conditions at the origin for the radial part of the real functions  $\{\xi_{slmP}(t, r)\}$  [6]. In the first term of eqn.(4), one must understand that the limit  $R_\infty \rightarrow \infty$  is implied.

Equation (3) can be interpreted as giving a ‘coordinate’-representation amplitude for a set of final field configurations  $\{a_{sklmP}\}$  labelled by ‘quantum’ numbers  $\{sklmP\}$ , given that on the hypersurface  $\Sigma_I$   $\{t = 0\}$  the perturbations vanished, i.e., the metric and background matter were spherically symmetric. Hence  $|\Psi[\{a_{sklmP}\}; T]|^2$  is a (conditional) probability density of finding the field in a set of final configurations  $\{a_{sklmP}\}$  at asymptotic proper time  $T$ .

To proceed further, let us discretise the frequency integral in eqn.(4), with  $\{k_j > 0\}$  the eigenfrequencies of the final radiation located in a spatial volume on  $\Sigma_F$  — the spatial momenta. An explicit expression for the  $\{k_j\}$  will not be needed, although the continuum limit is recovered in the limit  $R_\infty \rightarrow \infty$ . Denoting  $j = \{slmP\}$ , we find that eqn.(3) can be written as (see also Appendix A)

$$\begin{aligned}\Psi[\{A_j\}; T] &= \hat{N} e^{-i\frac{1}{2}M_I T} \prod_j \frac{1}{2i \sin(k_j T)} \exp\left(\frac{i}{2} \Delta k_j k_j |A_j|^2 \cot(k_j T)\right) \\ &= \hat{N} e^{-i\frac{1}{2}M_I T} e^{-\frac{1}{2} \sum_j \Delta k_j k_j |A_j|^2} \prod_j \sum_{n=0}^{\infty} e^{-2iE_n T} L_n(k_j \Delta k_j |A_j|^2),\end{aligned}\quad (7)$$

where

$$|A_j|^2 = 2(-1)^s w_s \frac{(l-s)!}{(l+s)!} |z_j|^2 |a_j + (-1)^s P a_{s,-k_j l m P}|^2, \quad (8)$$

and  $E_n = k_j(n + \frac{1}{2})$  is the quantum energy of the linear harmonic oscillator. Note the dependence of the quantum amplitude on  $|A_j|$ ; that is, it is ‘spherically symmetric’. The  $\{L_n\}$  are Laguerre polynomials with

$$L_n^{(m-n)}(x) = \sum_{p=0}^n \binom{m}{n-p} \frac{(-x)^p}{p!} \quad (9)$$

the associated Laguerre polynomials, and  $L_n^{(0)}(x) = L_n(x)$  [8]. The  $\{L_n(x)\}$  satisfy the completeness relation

$$\sum_{n=0}^{\infty} e^{-\frac{1}{2}x} L_n(x) e^{-\frac{1}{2}y} L_n(y) = \delta(x-y). \quad (10)$$

The Laguerre polynomials  $\{L_n(|z|^2)\}$ , where  $z = x + iy$ , cannot be written as a product of two decoupled wave functions of  $x$  and  $y$  in an excited state (due to pair correlations [22] in a quantum interpretation), but we can write [9]

$$L_n(x^2 + y^2) = \frac{(-1)^n}{2^{2n} n!} \sum_{p=0}^n \binom{n}{p} H_{2p}(x) H_{2n-2p}(y). \quad (11)$$

The wave functional  $\Psi[\{A_j\}; T]$  is proportional to a product of delta functions when  $k_j T = p\pi$  ( $p$  being an integer), following from the completeness relation eqn.(10) – see Appendix A. At these focal points, the density of paths – effectively the prefactor in the first line of eqn.(7) – diverges.

## 4 Coherent States

The Schrödinger picture wave functions

$$\Psi_{nj}(x_j, T) = \frac{N}{\pi} e^{-\frac{1}{2}x_j} e^{-2iE_n T} L_n(x_j), \quad (12)$$

where  $x_j = k_j \Delta k_j |A_j|^2$ , have a strong connection with the exact solution of the forced-harmonic-oscillator problem [10]. In this theory, one considers a 1-d harmonic

oscillator with mass  $\mu$  and frequency  $\omega$ , which is acted on by an external force  $F(t)$ . In this case, the Hamiltonian has the form

$$H = \frac{1}{2}(p^2 + q^2) + qF(t). \quad (13)$$

Suppose that for  $t_0 < t < T$  this force is non-vanishing so that asymptotic states are free oscillator states. We want to calculate the probability amplitude  $A_{km}$  to make a transition from the free oscillator state  $|m\rangle$  (with  $m$  particles) at time  $t' < t_0$ , before the force begins to act, to the free oscillator state  $|k\rangle$  at time  $t > T$ , after the force has ceased. Set

$$z = \frac{|\beta|^2}{2\mu\omega}, \quad (14)$$

$$\beta = \int_{t_0}^T dt F(t) e^{-i\omega t}, \quad (15)$$

effectively the ‘Fourier’ transform of the force. It has been shown that [11] – [13] ( $m \geq k$ )

$$A_{km} = e^{i\lambda} e^{-\frac{1}{2}z} \left( \frac{k!}{m!} \right)^{\frac{1}{2}} \left( \frac{i\beta}{\sqrt{2\mu\omega}} \right)^{m-k} L_k^{(m-k)}(z), \quad (16)$$

where  $\lambda$  is a real phase and  $A_{km}$  is symmetric in  $k$  and  $m$ . An adiabatic approximation ( $z \ll 1$ ) indicates that a state which begins as  $|m\rangle$  must end up in the same state at late times after the time-dependent force has been removed. Then

$$A_{kk} = e^{i\lambda} e^{-\frac{1}{2}z} L_k(z). \quad (17)$$

The probability that there be no change in the number of particles is therefore  $|A_{kk}|^2 = e^{-z} (L_k(z))^2$ . Apart from the introduction of mode labels  $\{j\}$  and a necessary reinterpretation for  $z$ , these amplitudes are effectively the semi-classical wave functions Eqn.(12) derived from our boundary-value problem.

A brief derivation of eqn.(16) in the context of the coherent-state representation [13] will be useful for the calculations in Section 4.1. Coherent states  $|\alpha\rangle$  can be regarded as displaced vacuum states [14]:

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad (18)$$

where  $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$  is a unitary displacement operator ( $D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha)$ ) and the states  $|\alpha\rangle$  are eigenstates of the annihilation operator  $a$  with complex eigenvalue  $\alpha$ . They are the closest states to classical states in that they attain the minimum demanded by the uncertainty principle. Thus, coherent states may also be useful when one has a large number of particles, as in the classical limit, and when one has some phase information about the state. Coherent states form an over-complete set, but are not orthogonal. In terms of Fock number states  $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$  [13],

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (19)$$

The coherent state corresponding to  $\alpha = 0$  is the unique state of the oscillator, namely, the Fock state  $|n\rangle$  with  $n = 0$ . Thus, if the system started in a vacuum state, the amplitude to then find it in a coherent state  $|\alpha\rangle$  is

$$\langle 0|\alpha\rangle = \langle 0|D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}, \quad (20)$$

up to a phase.

From the properties of displacement operators [14], then

$$\begin{aligned} D(\xi)|\alpha\rangle &= D(\xi)D(\alpha)|0\rangle \\ &= D(\xi + \alpha)|0\rangle e^{\frac{1}{2}(\xi\alpha^* - \xi^*\alpha)} \\ &= |\xi + \alpha\rangle e^{\frac{1}{2}(\xi\alpha^* - \xi^*\alpha)}. \end{aligned} \quad (21)$$

For later reference, one can show that

$$D^\dagger(\alpha)D(\mu)D(\alpha) = D(\mu)e^{\alpha^*\mu - \alpha\mu^*}. \quad (22)$$

Using eqns.(19) and (21),

$$\langle m|D(\xi)|\alpha\rangle = \frac{1}{\sqrt{m!}}(\xi + \alpha)^m e^{-\frac{1}{2}(|\alpha|^2 + |\xi|^2 + 2\xi^*\alpha)}. \quad (23)$$

Also from eqn.(19),

$$\langle m|D(\xi)|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle m|D(\xi)|n\rangle. \quad (24)$$

Equating eqns.(23) and (24) then gives

$$(1 + y)^m e^{-y|\xi|^2} = e^{\frac{1}{2}|\xi|^2} \sum_{n=0}^{\infty} \sqrt{\frac{m!}{n!}} \xi^{n-m} y^n \langle m|D(\xi)|n\rangle. \quad (25)$$

The generating function for the associated Laguerre polynomials [8]

$$(1 + y)^m e^{-yx} = \sum_{n=0}^{\infty} L_n^{(m-n)}(x) y^n, \quad |y| < 1, \quad (26)$$

then implies that the matrix element between initial and final states can be written as

$$\langle m|D(\xi)|n\rangle = \left(\frac{n!}{m!}\right)^{\frac{1}{2}} \xi^{m-n} e^{-\frac{1}{2}|\xi|^2} L_n^{(m-n)}(|\xi|^2), \quad (27)$$

which is eqn.(16) up to an unimportant phase factor. Interchanging  $m$  and  $n$  gives the same result.

#### 4.1 Generalised Coherent States

There is an interpretation for these amplitudes in terms of generalised coherent states  $|n, \alpha\rangle$  of the harmonic oscillator [12]. Defining

$$|n, \alpha\rangle = e^{-iE_n t} D(\alpha(t)) |n\rangle, \quad (28)$$

then in the Fock representation

$$|n, \alpha \rangle = \sum_{m=0}^{\infty} \langle m | D(\alpha(0)) | n \rangle | m \rangle e^{-iE_m t}. \quad (29)$$

For generalised coherent states (see eqn.(33) below), the ground state ( $n = 0$ ) is a coherent state and not a vacuum state. Generalised coherent states are to the coherent states what the Fock states  $|n \rangle$  are to the vacuum state, that is, excited coherent states. In addition, one has

$$I = \frac{1}{\pi} \int d^2\alpha |n, \alpha \rangle \langle n, \alpha|, \quad (30)$$

$$\langle n, \beta | n, \alpha \rangle = L_n(|\alpha - \beta|^2) e^{\beta^* \alpha - \frac{1}{2}(|\alpha|^2 + |\beta|^2)}, \quad (31)$$

$$\langle n, \beta | \psi \rangle = \frac{e^{-\frac{1}{2}|\beta|^2}}{\pi} \int d^2\alpha L_n(|\alpha - \beta|^2) e^{\beta^* \alpha - \frac{1}{2}|\alpha|^2} \langle n, \alpha | \psi \rangle, \quad (32)$$

for an arbitrary state  $|\psi \rangle$ , where

$$\frac{1}{\pi} \int d^2\alpha = \int d(Re\alpha) d(Im\alpha),$$

and  $I$  is the identity operator. From eqn.(31) with  $\beta = 0$ , then

$$\langle n, 0 | n, \alpha \rangle \equiv \langle n | n, \alpha \rangle = e^{-\frac{1}{2}|\alpha|^2} L_n(|\alpha|^2), \quad (33)$$

again giving eqn.(12) up to a phase and normalisation. The initial state may be seen not as a vacuum state, but as a Fock state, and the final state as a generalised coherent state with the same  $n$ .

An interesting interpretation for the amplitudes eqn.(27) is that these are the matrix elements for the transition from the state  $|k \rangle$  to  $|m \rangle$  under the influence of a gravitational wave [15], with the force  $F(t)$  proportional to the Riemann curvature tensor component  $R_{xtxt}(t)$ :

$$F(t) = \mu c^2 l R_{xtxt}(t) = -\frac{1}{2} \mu c^2 l \partial_t^2 h_{xx}^{TT}, \quad (34)$$

where  $l$  is the distance between two particles each of mass  $\frac{1}{2}\mu$  along the  $x$ -axis,  $h_{xx}^{TT}$  is the transverse traceless gravitational wave component [16] and  $x$  is the change in separation of the masses. In the context of black-hole evaporation, one can understand the time-dependent force as being active during the time-dependent phase of the collapse to a black hole and its subsequent complete evaporation. The duration of the ‘force’, therefore, is comparable to the lifetime of the black hole. A space-time would begin in an (almost) static state in the far past, where the perturbation modes were effectively free, pass through an intermediate time-dependent phase, to end up in a static or, rather, quasi-stationary state in the far future, where again the perturbation modes are free. In this ‘sandwich’-space-time picture, particles are created by the space-time curvature, or non-adiabatic behaviour of the fields in a time-dependent metric. What the above calculations indicate is an explicit mathematical connection between the theory of forced harmonic oscillators and certain amplitudes relating to the dynamical evolution of black holes.

## 5 Squeezed-State Formalism

In this section, we see how by rotating the proper time at infinity  $T$  into the complex plane, and with spherically symmetric initial matter and gravitational fields, one obtains a quantum-mechanical squeezed-state interpretation for the final state of the Hawking radiation.

Grishchuk and Sidorov [4] were the first to explicitly formulate particle creation in strong gravitational fields in terms of squeezed-states, although the formalism does appear in Parker's original paper on cosmological particle production [18]. In ref. [4], it was shown that relic gravitons (as well as other perturbations), created from zero-point quantum fluctuations as the universe evolves, should now be in a strongly squeezed state. Squeezing is just the quantum process of parametric amplification. In ref. [24] however, the claim was made that there was no new physics in employing the squeezed-state formalism.

### 5.1 Squeezed States – Introduction

A general one-mode squeezed state (or squeezed coherent state) is defined as [17]

$$|\alpha, z\rangle = S_1(r, \varphi)D(\alpha)|0\rangle = S_1(z)D(\alpha)|0\rangle = S_1(z)|\alpha\rangle, \quad (35)$$

where  $D(\alpha)$  is the single-mode displacement operator,  $S_1(r, \varphi)$  is the unitary squeezing operator ( $S_1^\dagger(z)S_1(z) = S_1(z)S_1^\dagger(z) = 1$ ) for  $|\alpha, z\rangle$ :

$$S_1(r, \varphi) \equiv S_1(z) = e^{\frac{1}{2}(za^2 - z^*a^{\dagger 2})}, \quad (36)$$

where  $z = re^{-2i\varphi}$ . Further,  $S_1(r, \varphi)S_1(r', \varphi) = S_1(r + r', \varphi)$ . With  $a|\alpha\rangle = \alpha|\alpha\rangle$ , then setting  $b = S_1aS_1^\dagger$  implies that  $b|\alpha, z\rangle = \alpha|\alpha, z\rangle$  and  $b = a \cosh r + a^\dagger e^{2i\varphi} \sinh r$ . The state  $|\alpha, z\rangle$  is a Gaussian packet displaced from the origin in position and momentum space. While the real squeezing parameter  $r \geq 0$  determines the magnitude of the squeezing, the squeezing angle  $\varphi$  ( $-\frac{\pi}{2} < \varphi \leq \frac{\pi}{2}$ ), gives the distribution of the squeezing between conjugate variables. The operators  $a$  and  $a^\dagger$  are annihilation and creation operators, respectively. The squeezed vacuum state occurs when  $\alpha = 0$ :

$$|z\rangle \equiv |0, z\rangle = S_1(z)|0\rangle. \quad (37)$$

The high-squeezing limit corresponds to  $r \gg 1$ , where the state  $|z\rangle$  is highly localised in momentum space. The state with  $r = 0$  is the ground state.

Consider the amplitude  $A = \langle \alpha, z | D(\mu) | \alpha, z \rangle = \langle z | D^\dagger(\alpha) D(\mu) D(\alpha) | z \rangle$ . One can use eqn.(22) to show that  $A = \langle z | D(\mu) | z \rangle e^{2i\text{Im}(\alpha^*\mu)} = \langle 0 | D(\eta) | 0 \rangle e^{2i\text{Im}(\alpha^*\mu)}$ , where  $\eta = \mu \cosh r - \mu^* e^{2i\varphi} \sinh r$ , and then use  $\mu = |\mu|e^{i\phi}$  to show that [15]

$$|A|^2 = e^{-|\eta|^2} = e^{-|\mu|^2 [\cosh 2r - \cos 2(\phi + \varphi) \sinh 2r]}, \quad (38)$$

which is independent of the initial displacement  $\alpha$ . Equation (38) will appear when we match our space-like hypersurface approach with the method of Bogoliubov coefficients [6].

Single-mode squeezed operators do not conserve momentum since they describe the creation of particle pairs with momentum  $k$ . Two-mode squeezed operators,

however, describe the creation and annihilation of two particles (waves) with equal and opposite momenta. A two-mode squeeze operator has the form [29]

$$S_2(r, \varphi) = e^{r(e^{-2i\varphi}a_+a_- - e^{2i\varphi}a_+^\dagger a_-^\dagger)}, \quad S_2^\dagger = S_2^{-1} = S_2(r, \varphi + \frac{\pi}{2}), \quad (39)$$

where  $a_\pm, a_\pm^\dagger$  are annihilation and creation operators for the two modes, respectively.

One can also introduce a unitary rotation operator

$$R(\theta) = e^{-i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)}, \quad (40)$$

with  $0 \leq \theta \leq 2\pi$ ,  $R(\theta)R(\theta') = R(\theta + \theta')$ ,  $R|0\rangle = |0\rangle$ ,  $RaR^\dagger = e^{i\theta}a$ . However, rotation does not influence particle creation [29].

The variances of conjugate operators  $\hat{p}$  and  $\hat{q}$  are given by  $\Delta\hat{q} = \hat{q} - \langle \hat{q} \rangle$ ,  $\Delta\hat{p} = \hat{p} - \langle \hat{p} \rangle$ . In the squeezing formalism,  $\Delta\hat{q}$  and  $\Delta\hat{p}$  differ greatly, while they are equal and the minimum possible for coherent states, deemed to be the most classical of quantum states. The name ‘squeezed’ refers to the fact that the variance of one variable in a conjugate pair can go below the minimum allowed by the uncertainty principle (the squeezed variable), while the variance of the conjugate variable can exceed the minimum value allowed (the superfluctuant variable) [22] [26] [27]. The superfluctuant variable is amplified by the squeezing process, and so it is possible to observe macroscopically, while the subfluctuant variable is squeezed and becomes unobservable. In particle production, for example, the number operator is a superfluctuant variable, while the phase is squeezed. Squeezed states are essentially purely quantum-mechanical in origin.

Hawking radiation in the squeezed-state representation was first discussed in ref. [4]. The squeeze parameter  $r_j$  was related to frequency and the black-hole mass through  $\tanh r_j = e^{-4\pi\omega_j M}$ ,  $\theta_j = \varphi_j$  (see also refs. [21] and [32]). The vacuum quantum state in a black-hole space-time for each mode is a two-mode squeezed vacuum.

## 5.2 Analytic Continuation and the Large-Squeezing Limit

Consider the Schrödinger picture quantum state eqn.(7) associated with the final hypersurface  $\Sigma_F$ , and define

$$\begin{aligned} \Phi[\{A_j\}; T] &= \prod_j 2i \sin(k_j T) \Psi[\{A_j\}; T] = \hat{N} e^{-i\frac{1}{2}M_I T} \prod_j \exp \left[ \frac{i}{2} \Delta k_j k_j |A_j|^2 \cot(k_j T) \right] \\ &\equiv \hat{N} \exp \left( iS^{(cl)}[\{A_j\}; T] \right). \end{aligned} \quad (41)$$

In order for our two-surface boundary-value problem to be well posed, we argue that  $T$  must be a complex quantity. This is because for real  $T$ , the ‘sum’ in eqn.(41) diverges due to the simple poles on the real-frequency axis at  $k_j = \sigma_n = \frac{n\pi}{T}$ ,  $n = 1, 2, \dots$ , assuming that  $k_j |A_j|^2$  remains finite and non-zero near  $k_j = \sigma_n$ . At the frequencies  $\{\sigma_n\}$ , there may be an infinite number of classical paths (solutions) joining the initial and final data, or perhaps none at all. Should a solution exist, it may not depend smoothly on the boundary values. These are features of ill-posed boundary-value problems. Yet, if  $T$  is deformed into the complex plane, the poles

along the real- $k_j$  axis would be displaced into the complex- $k$  plane. The complex boundary-value problem itself guarantees that the classical solution is complex analytic (strong ellipticity). One then has a reasonable expectation that there will now be a smooth, unique classical solution of the field equations joining the initial and final data. Essentially, this is the  $i\varepsilon$  prescription of quantum-field theory. In general, therefore, our background metric  $\gamma_{\mu\nu}$  (and hence  $g_{\mu\nu}$ ) is not a real Lorentzian metric but perhaps a strongly-elliptic metric. This may permit the path integral over the boundary data to be well-defined. Even for a complexified  $T$ , one still has a time,  $|T|$ , whose ‘gradient’ is everywhere future-directed and time-like. By considering the dimensionless quantity  $T/2M_I$ , one also sees that analytic continuation of the complex quantity  $T$  is equivalent to adding a small positive imaginary part to the mass, as is the case in the Damour-Ruffini tunnelling approach to black-hole evaporation [19].

In terms of the late stages of black-hole evolution, we may use analytic continuation to tackle the crucial question of whether future null infinity  $\mathcal{I}^+$  is a Cauchy hypersurface for space-time. If we model the evaporation process with a classical Lorentzian space-time metric, it is well-known that a momentarily-naked singularity is inevitable [20]. Cosmic censorship, a postulate of classical general relativity, is seemingly transcended through the quantum-mechanical evaporation of the black hole. Consequently, a Bogoliubov transformation between initial and final modes would be forbidden. There would then be no one-to-one correspondence between pre-collapse configurations and post-evaporation configurations, no S-matrix and, therefore, there will be evolution from a pure initial state to a final density matrix. Our reasoning is that a non-singular asymptotically-flat initial state on a suitable Cauchy hypersurface can never evolve to a final (even momentarily) naked singularity if we complexify the proper-time separation at spatial infinity, so that the future is entirely predictable from the past. If measurements at  $\mathcal{I}^+$  are insufficient to determine the state at  $\mathcal{I}^-$ , one needs boundary conditions – or another set of ‘coordinates’ – on the Cauchy horizon which forms after the black hole disappears. Boundary conditions along the Cauchy horizon represent constraints on the initial conditions as configurations on a final Cauchy hypersurface must be sufficient to determine the state on an initial Cauchy hypersurface. That there is no singularity or edge to space-time is a necessity if the path integral for quantum gravity is to be a no-boundary wave function [3].

In the presence of a time-independent Schwarzschild black hole, say, one commonly ‘Wick’ rotates the real time  $t$   $90^\circ$  into the lower half-plane ( $t = -i\tau$ ) [1]. One now has a Euclidean time separation at spatial infinity and the space-time boundary is real and positive-definite. Where there is a future event horizon, one has the additional boundary component at the axis  $r = 2M_0$ . In the Euclidean *régime*, a conical singularity is present at  $r = 2M_0$  unless the Euclidean time is periodically identified with period  $8\pi M_0$  [2]. Thus, the coordinate singularity at  $r = 2M_0$  as well as the curvature singularity at  $r = 0$  is avoided, since  $r > 2M_0$  for Euclidean signature. The manifold for which  $r \geq 2M_0$  and  $0 \leq \tau \leq 8\pi M_0$  is the real Euclidean section of the Schwarzschild solution. Boundary conditions for physical fields then need not be specified at the curvature singularity, which does not lie on the Euclidean section, and these fields are also regular at the horizon. This was

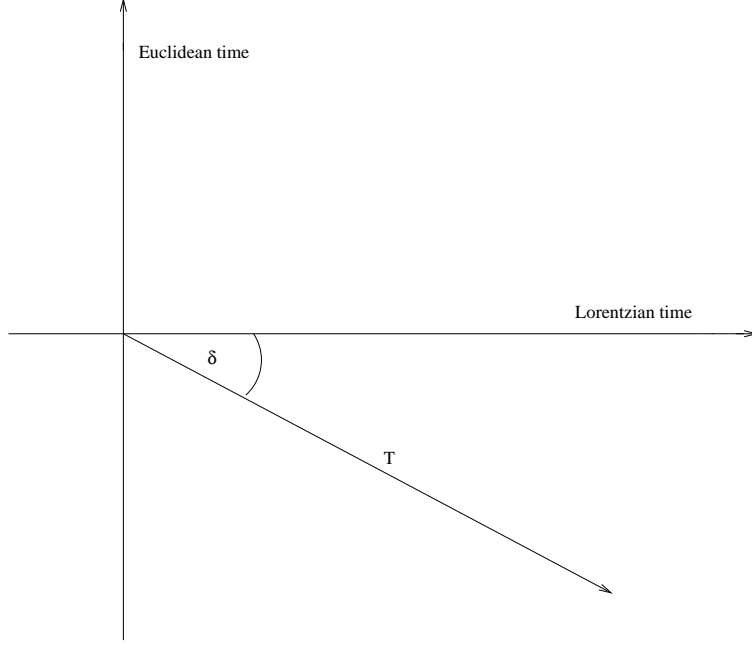


Figure 1: Infinitesimal deformation of asymptotic proper-time separation by an angle  $\delta$  into the lower complex plane.

interpreted as quantum cosmic censorship, and with the regularity at  $r = 2M_0$ , one is seemingly summing over all possible configurations which lie inside the black hole  $r < 2M_0$ .

Hence, let us write (see fig.1)

$$T = |T|e^{-i\delta}, \quad 0 \leq \delta < \frac{\pi}{2}. \quad (42)$$

Below we shall consider the case of infinitesimal  $\delta$ . Then, from eqns.(41) and (42)

$$\begin{aligned} \Phi[\{A_j\}; T] &= \Phi[\{A_j\}; |T|, \delta] \\ &= \hat{N} e^{-\frac{1}{2}iM_I|T| \cos \delta} e^{-\frac{1}{2}M_I|T| \sin \delta} \prod_j \exp \left[ -\frac{1}{2} \Delta k_j k_j |A_j|^2 \coth(k_j |T| \sin \delta - i\varphi_j(|T|, \delta)) \right] \\ &= \hat{N} e^{-\frac{1}{2}iM_I|T| \cos \delta} e^{-\frac{1}{2}M_I|T| \sin \delta} \prod_j \exp \left( -\frac{1}{2} [\Omega_{Rj} + i\Omega_{Ij}] \Delta k_j k_j |A_j|^2 \right), \end{aligned} \quad (43)$$

where

$$\varphi_j(|T|, \delta) = -k_j |T| \cos \delta, \quad (44)$$

and

$$\Omega_{Rj}(|T|, \delta) = \frac{\sinh(2k_j |T| \sin \delta)}{2[\cosh^2(k_j |T| \sin \delta) - \cos^2 \varphi_j]}, \quad \Omega_{Ij}(|T|, \delta) = -\frac{\sin 2\varphi_j}{2[\cosh^2(k_j |T| \sin \delta) - \cos^2 \varphi_j]}. \quad (45)$$

One can also write eqn.(43) as

$$\Phi[\{A_j\}; |T|, \delta] = \hat{N} e^{-\frac{1}{2}iM_I|T|\cos\delta} e^{-\frac{1}{2}M_I|T|\sin\delta} \prod_j \exp \left[ -\frac{1}{2} \Delta k_j k_j \left( \frac{1 + e^{2i\varphi_j} \tanh r_j}{1 - e^{2i\varphi_j} \tanh r_j} \right) |A_j|^2 \right], \quad (46)$$

where we have formally set

$$\tanh r_j(|T|, \delta) = e^{-2k_j|T|\sin\delta}. \quad (47)$$

Therefore,

$$e^{-2r_j} = \tanh(k_j|T|\sin\delta). \quad (48)$$

We recognise eqn.(46) as the coordinate-space representation of a quantum-mechanical squeezed state [30] [32], with  $r_j(|T|, \delta)$  the squeeze parameter and  $\varphi_j(|T|, \delta)$  the squeeze angle. The evolution of the squeezed state is taken into account by the  $|T|$  dependence in  $r_j$  and  $\varphi_j$ , which are in general complicated functions of time. Equation (44) is more familiar in the limit of infinitesimal  $\delta$ , whence neglecting  $O(\delta^2)$  terms,  $\varphi_j(|T|, \delta \ll \frac{\pi}{2}) \simeq -k_j|T|$ , corresponding to free evolution. The high-frequency limit  $k_j|T| \gg 1$  corresponds to  $r_j \rightarrow 0$  for  $\sin\delta \neq 0$ .

Computing the probability density  $|\Phi[\{A_j\}; |T|, \delta]|^2$ , we find for small  $\delta$

$$|\Phi[\{A_j\}; |T|, \delta]|^2 = |\hat{N}|^2 e^{-M_I|T|\delta} \prod_j \exp \left[ -\frac{\coth \varepsilon_j}{1 + \frac{\sin^2(k_j|T|)}{\sinh^2 \varepsilon_j}} \Delta k_j k_j |A_j|^2 \right], \quad (49)$$

where  $0 < \varepsilon_j \equiv k_j|T|\delta \ll 1$ , i.e.,  $0 \leq \delta \ll (\tilde{k}_j \tilde{T})^{-1}$ , with  $\tilde{k}_j = 2M_I k_j$ ,  $\tilde{T} = \frac{|T|}{2M_I}$ . Then, from eqn.(48)

$$\varepsilon_j \simeq e^{-2r_j}, \quad \varepsilon_j \ll 1, \quad (50)$$

corresponding to  $r_j \gg 1$ , which is the high-squeezing limit.

In the squeezed-state formalism, this limit is deemed to be the classical limit when the average number of particles in the final state is large:  $\langle N_j \rangle = \sinh^2 r_j \simeq \frac{1}{4} e^{2r_j}$  for  $r_j \gg 1$ . Here, the limit of infinitesimal  $\delta$  is also a quantum-to-classical transition, from a Euclidean ( $\delta \sim \frac{\pi}{2}$ ) theory, where there is no concept of time, to a semi-classical Lorentzian theory, with infinitesimal  $\delta$ , where the notion of a classical time parameter appears. Such issues are related to the quantum cosmology program of Hartle and Hawking [3] in the emergence of the classical universe from a smooth Euclidean (quantum) origin, where there is no cosmological-singularity.

Indeed, the semi-classicality of the state eqn.(41) is intimately related to the high-squeezing limit [24]. If we just consider  $M_I$  as a fixed parameter in the theory and not a functional of the final field configurations  $\{x_j\}$ , then the WKB condition is met when

$$\left| \frac{\Omega_{Ij}}{\Omega_{Rj}} \right| = \left| \frac{\sin 2\varphi_j}{\sinh(2k_j|T|\sin\delta)} \right| = |\sin 2\varphi_j \sinh 2r_j| \gg 1, \quad (51)$$

which is satisfied in the high-squeezing limit  $\varepsilon_j \ll 1$  even if  $\sin 2\varphi_j = 0$ . The final state of the remnant Hawking flux, therefore, becomes more classical in the WKB sense in the limit  $\delta \rightarrow 0$ . In this limit, one can effectively consider the

final perturbations as being represented by a classical probability distribution function [4] [24] [40]. As in the inflationary scenario, the perturbations on the black-hole background space-time, which had a quantum-mechanical origin, cannot be distinguished from classical stochastic perturbations, without the need of an environment for decoherence. It is the field amplitudes which one observes, and these amplitudes have a classical phase-space distribution. The squeezed nature of the final quantum state for the stochastic gravitational-wave background produced by inflation is not stationary [4].

The initial conditions for the perturbations in the black-hole case also have an inflationary analogue. In cosmology, the assumption is that at some early ‘time’ just prior to inflation, the modes are in their adiabatic ground state. This originates from the assumption that the universe was in a maximally-symmetric state at some time in the past [3]. A similar assumption obtains in our black-hole case, where we assumed that the initial perturbations were very weak so that the initial matter and its gravitational field were spherically symmetric.

We assume that  $\hat{T} \gg 1$ ; that is, we observe the black hole at infinity at times much greater than its collapse timescale, which is of order  $\pi M_I$  [16]. If  $\tilde{k}_j$  is moderately large, then for small  $\varepsilon_j$  (keeping  $O(\varepsilon_j^2)$  terms)

$$\begin{aligned} |\Phi[\{A_j\}; |T|, \delta]|^2 &= |\hat{N}|^2 e^{-M_I |T| \delta} \prod_j \exp \left[ -\frac{\varepsilon_j}{\varepsilon_j^2 + \sin^2(k_j |T|)} \Delta k_j k_j |A_j|^2 \right] \\ \varepsilon_j \xrightarrow{=} 0 & \quad |\hat{N}|^2 \prod_j \exp \left[ -\Delta k_j k_j |A_j|^2 \sum_{n=1}^{\infty} \Delta \omega_n \delta(k_j - \omega_n) \right], \end{aligned} \quad (52)$$

where  $\omega_n = \frac{n\pi}{|T|}$ ,  $\Delta \omega_n = \frac{\pi}{|T|}$  and we used  $k_j > 0$  with  $k_j |A_j|^2 \rightarrow 0$  as  $k_j \rightarrow 0$  to convert the sum  $\sum_{n=-\infty}^{\infty}$  into  $\sum_{n=1}^{\infty}$ . Hence, assuming the interchange of  $j$  and  $n$  sums in the continuum limit for the  $\{k_j\}$  frequencies is valid, then

$$|\Phi[\{A_j\}; |T|, \delta \rightarrow 0]|^2 = |\hat{N}|^2 \prod_{slmP} \prod_{n=1}^{\infty} e^{-\Delta \omega_n \omega_n |A_{snlmp}|^2}, \quad (53)$$

so the frequencies  $\{\omega_n\}$  dominate the final state. This is the same as in ref. [6] where we used contour integration to obtain the probability density. Equation (52) describes a Gaussian non-stationary process in that the variance is an oscillatory function of the asymptotic proper time. Rather than travelling waves, one is now dealing with standing bosonic waves, where the amplitudes for left- and right-moving waves are large and almost equal, similar to the cosmological scenario [4] – see eqn.(1). These standing waves imply a correlation between particles with opposite frequencies (and azimuthal angular momentum  $m$ ) in the final state, indicative of a pure state. We suggest that the discrete frequencies  $\{\omega_n\}$  are, in principle, observable feature of the relic Hawking radiation, either directly or indirectly.

The presence of the delta function in eqn.(52) indicates that in the high-squeezing limit, the random variable  $\varphi_j$  associated with the final state is squeezed to discrete values independently of the quantum numbers  $\{slmP\}$ . Note that it is only the squeeze phases  $\{\varphi_j\}$  of the perturbations which are fixed and correlated in the high-squeezing limit. There are phases in our theory, however, which we deem to

have a random and even distribution; namely, the phases associated with the initial state of the fluctuations incoming from  $\mathcal{I}^-$ , and the phases associated with the spatial field distribution labelled by  $\{a_{sklmP}\}$ . The final field amplitudes for the remnant Hawking radiation reaching the null hypersurface  $\mathcal{I}^+$   $\{b_j\}$  can be matched on  $\Sigma_F$  with the amplitudes  $\{a_j\}$  associated with the space-like hypersurface  $\Sigma_F$  [6]. The  $\{b_j\}$  are in turn connected, through a Bogoliubov transformation, with amplitudes  $\{c_{j'}\}$  for travelling waves incoming from  $\mathcal{I}^-$ . A positive-definite probability distribution can be obtained by averaging over the random phase of the  $\{c_{j'}\}$  fluctuations. Therefore, it is only the phases  $\{\varphi_j\}$  associated with the standing waves which do not have a random character.

The normalisation factor  $|\hat{N}|^2$  is determined by integrating over the dimensionless variables  $\{y_j\}$ , where  $x_j = \Delta k_j k_j |A_j|^2 \equiv M_I^2 y_j$ , so that the sum of all probabilities of all possible configurations  $\{y_j\}$  is unity. Treating the initial ADM mass  $M_I$  as a fixed parameter rather than as a functional of  $\{x_j\}$ , we find that

$$\begin{aligned} |\hat{N}|^2 &= \prod_j \frac{M_I^2 \coth \varepsilon_j}{1 + \frac{\sin^2(k_j |T|)}{\sinh^2 \varepsilon_j}} \\ &= \prod_j [\cosh 2r_j - \cos 2\varphi_j \sinh 2r_j]^{-1} M_I^2. \end{aligned} \quad (54)$$

In fact, in the high-squeezing limit  $\delta \rightarrow 0$ , final states which differ by a coordinate-dependent phase  $e^{-\frac{1}{2}iM_I T}$  are deemed to be physically equivalent. If we considered the ADM mass as a functional of  $\{x_j\}$ , then the normalisation factor would be altered for finite  $\delta$ , as would the value of the entropy – see below. Thus, there is an ambiguity in the normalisation factor. In addition, this is related to the ambiguity in the form of the entropy in the high-squeezing limit with and without a surface term included in the Lagrangian [30]. In the limit  $\varepsilon_j \rightarrow 0$ , then

$$|\Phi[\{A_j\}; |T|, \delta \rightarrow 0]|^2 = \left[ \prod_j M_I^2 \sum_{n=1}^{\infty} \Delta \omega_n \delta(\omega_n - k_j) \right] \prod_{slmP} \prod_{n=1}^{\infty} e^{-\Delta \omega_n \omega_n |A_{slmP}|^2}. \quad (55)$$

This equation is reminiscent of the Wigner function corresponding to the vacuum state in the limit of infinite squeezing [40], giving the classical ‘trajectories’ of the system. The Wigner function cannot in general be interpreted as a classical probability density for finite squeezing, that is, finite  $\delta$ , except for Gaussian states.

A consequence of the high-squeezing behaviour is that the variance in the amplitudes  $\{x_j\}$  is large, so that there are large statistical deviations of the observable power spectrum from its expected value. This is just a manifestation of the uncertainty principle. Indeed, with respect to the first expression in eqn.(52), and using eqn.(54), we find that

$$\begin{aligned} \langle y_j \rangle_{|\Phi|^2} \quad \varepsilon_j \rightarrow 0 &= \left[ M_I^2 \sum_{n=1}^{\infty} \Delta \omega_n \delta(\omega_n - k_j) \right]^{-1} \\ &\equiv [\rho_j(M_I)]^{-1}, \end{aligned} \quad (56)$$

where  $\langle \rangle_{|\Phi|^2}$  denotes the expectation value with respect to the probability density  $|\Phi|^2$ . It is reasonable to suggest that  $\rho_j(M_I)$  represents a microcanonical density-of-states.

In inflationary cosmology, the oscillation phases of standing waves have fixed values, giving rise to zeros in the power spectrum, which are characteristic of the CMBR. In this case, the power spectrum of the cosmological perturbations in the present universe is not a smooth function of frequency. The standing-wave pattern, due to squeezing, induces oscillations in the power spectrum. This in turn produces Sakharov oscillations [24][42], produced by metric and scalar perturbations, in the distribution of higher-order multipoles  $l(l+1)C_l$  of the angular correlation function for the temperature anisotropies [38] [39] in the CMBR for all perturbations at a fixed time whose wavelength is of the order of or greater than the Hubble radius defined at this time. That is, the peaks and troughs of the angular power spectrum have a close relationship with the maxima and zeros of the metric power spectrum. However, for long wavelengths, the power spectrum is sufficiently smooth. Sakharov oscillations, therefore, exist as a result of the squeezed nature of the scalar and metric perturbations.

## 6 Entropy and Squeezing

There have been many accounts of how to determine the entropy generation in the squeezing formalism [22] – [30]. Hu and Pavon [31] were the first to associate entropy generation with the monotonic increase in the average particle number with time, induced by parametric amplification in a vacuum cosmological space-time. As squeezing is the quantum analogue of parametric amplification, one would expect that the squeezed-state formalism can compute entropy production. This is indeed the case, although, as with any entropy calculation, the nature of the coarse-graining must be specified. For squeezing, this is particularly relevant as squeezed evolution is unitary, i.e., there is no loss of information in principle in the evolution of the initial pure state to the final pure squeezed quantum state. How one chooses to measure the observables associated with the final squeezed state determines the entropy. One can reduce the final density matrix with respect to a Fock or coherent state basis [30], or use eigenstates of the superfluctuant variable [22] [26] [27]. In refs. [22] [26] [27], the loss of information comes from the increased dispersion of the superfluctuant operator.

In the classical limit of large average particle number, corresponding to the large-squeezing *régime*  $r_j \gg 1$ , a universal form for the entropy density growth  $\Delta S_j$  obtains for each mode [22] – [30]:

$$\Delta S_j \simeq 2r_j, \quad r_j \gg 1, \quad (57)$$

irrespective of the particular coarse-graining. In particular, on averaging over the squeezing angle  $\varphi_j$  of each Fourier mode, one obtains eqn.(57) [22] [23] [25] – [28]. One can determine whether coarse-graining with respect to the squeeze angle is appropriate from the prescription discussed in ref. [32]. Calculating the entropy  $S$  from eqn.(49) using eqn.(54), then

$$S = - \int \prod_j dy_j P(y_j) \ln P(y_j)$$

$$= 1 + \sum_j \ln \left[ M_I^{-2} (e^{2r_j} \sin^2 \varphi_j + e^{-2r_j} \cos^2 \varphi_j) \right]. \quad (58)$$

Thus, the entropy eqn.(58) comes from our ignorance of the final radiation configuration. The constant term is not important. In the high-squeezing limit

$$S \simeq 1 - \sum_j \ln M_I^2 + 2 \sum_j r_j + \sum_j \ln \sin^2 \varphi_j. \quad (59)$$

Evidently,  $\sin \varphi_j = 0$  for particular values of the frequency. Even if  $\sin \varphi_j \rightarrow 0$ , for  $r_j \rightarrow \infty$  and ignoring constant terms, then one may argue that  $\Delta S_j \simeq 2r_j$ . However, an alternative derivation keeping  $O(\varepsilon_j^2)$  terms gives from eqn.(54)

$$\begin{aligned} S &= 1 - \sum_j \ln \left[ M_I^2 \sum_{n=1}^{\infty} \Delta \omega_n \delta(\omega_n - k_j) \right] \\ &= 1 - \sum_j \ln \rho_j(M_I). \end{aligned} \quad (60)$$

How we take the high-squeezing limit is therefore important. The effect of the final term in eqn.(59) is to reduce the entropy from the maximal value given in eqn.(57). In the cosmological scenario, the primordial gravitational-wave background entropy is significantly smaller than eqn.(57) due to the information about the initial inflationary state of the universe manifesting itself in the primordial peaks in the multipole spectra of the CMB temperature anisotropy [43]. The presence of these peaks, therefore, is incompatible with a totally random squeezing angle, and so a coarse-graining with respect to  $\varphi_j$  is not possible. A similar conclusion may be argued in our case. We will comment on this in another paper [6], where we discuss the discreteness of the radiation frequencies in the context of Bekenstein's discrete event-horizon-area theory, and how information about the initial state may be contained in the spectral lines.

## 6.1 Classical Predictions

We now discuss how strong peaks in the wave function lead to definite predictions. In quantum cosmology, wave functions are commonly peaked about correlations between coordinates and momenta. In ref. [34], identification of such correlations came *via* the Wigner function. If the Wigner function  $W(p, q)$  factorised into a function of position  $q$  and a function of momentum  $p$ , then no correlation between  $q$  and  $p$  is predicted. When  $W(p, q)$  is peaked about some region in phase space  $p = f(q)$ , then the wave function predicts this particular correlation.

An alternative proposal for measuring correlations was given in ref. [35]. Here, projection onto coherent states, where momentum and position are equally known, as in classical theory, is employed for predicting classical correlations from a general Wigner function. It was shown that in the harmonic-oscillator case, the correlation between  $p$  and  $q$  was such that the Hamiltonian equalled the classical energy. We arrive at similar conclusions. In ref. [34], the predicted correlation was between the Hamiltonian equated to the quantum energy. The problem with this approach was that crude approximate Wigner functions bore no resemblance to the exact

Wigner functions. How Wigner functions appear in the context of our two-surface formalism will be considered in another paper.

If the quantum state  $\Psi$  is ‘sufficiently’ peaked about a region in the phase space, we observe correlations between the observables which characterise this region. Phase-space configurations for which  $\Psi$  is small are precluded and will not be observed. Where  $\Psi$  is neither small nor sufficiently peaked, no predictions can be made [36]. These conclusions are drawn from a new interpretation of quantum mechanics where wave functions are not associated with probabilities.

With the ADM mass  $M_I$  as a parameter, we look for predictions from the Heisenberg picture wave functional  $\Psi_{n_j}^{(H)}(x_j) = \frac{N}{\pi} e^{-\frac{1}{2}x_j} L_{n_j}(x_j)$ . Were we to restore our units, then the argument of the Laguerre polynomial would be proportional to  $\hbar^{-1}$ . Thus, in the limit of large argument, we use for large  $x$  [8]

$$L_k(x) \simeq \frac{(-x)^k}{k!}. \quad (61)$$

From a dimensional argument,  $x_j \propto \frac{M_I^2}{m_{pl}^2} \gg 1$ . In this case also, therefore, the approximation in eqn.(61) applies. One can now find the peak in the wave function as a function of  $x_j$  at

$$n_j = \frac{1}{2} k_j \Delta k_j |A_j|^2 \hbar^{-1}. \quad (62)$$

Considering the spin-0 case, for example, from eqn.(8) and eqn.(62)

$$n_j = 2\pi k_j \Delta k_j |z_j|^2 |a_j + (-1)^s P a_{s,-k_j l m P}|^2. \quad (63)$$

In a previous paper [6], we showed that for spin-0 perturbations

$$|b_j|^2 = 2\pi k_j |z_j|^2 |a_j + (-1)^s P a_{s,-k_j l m P}|^2, \quad (64)$$

where the  $\{b_j\}$  are Fourier amplitudes associated with the radiation reaching  $\mathcal{I}^+$ ; i.e., eqn.(64) matches the positive-frequency decomposition for massless spin-0 particles reaching  $\mathcal{I}^+$  (travelling waves) with our two-surface method for field configurations on the  $\{t = T\}$  hypersurface  $\Sigma_F$  (standing waves). Hence,  $n_j = \Delta k_j |b_j|^2$ . Thus we find that

$$\sum_j \hbar n_j k_j = \sum_j \Delta k_j k_j |b_j|^2 = M_I. \quad (65)$$

The right-hand side is just the final total energy in the massless spin-0 fluctuations, which equals the initial ADM mass  $M_I$ , as demanded by our boundary-value problem. The left-hand side is just the total energy of the radiated particles.

## 7 Conclusion

In this paper, we have illustrated many aspects of the two-surface formulation for linearised integer-spin fields propagating in an evaporating black-hole space-time. When the proper-time separation  $T$  at infinity between the initial and final hypersurfaces is deformed infinitesimally into the lower complex plane, and one has

spherically-symmetric initial fields, one obtains a quantum-mechanical squeezed-state formalism. The large squeezing limit is equivalent to the WKB limit and corresponds to an infinitesimal angle in the lower complex- $T$  plane. This we believe is related to the emergence of time in a semi-classical Lorentzian space-time from a timeless Euclidean *régime*. As the highly-squeezed final state is a pure state, our complexification technique has seemingly avoided the unpredictability associated with the momentarily-naked singularity, which is believed to be present prior to the complete disappearance of the black hole.

As in the cosmological scenario, we found that the bosonic perturbations on the black-hole background can be deemed to be a stochastic collection of standing waves rather than travelling waves in the high-squeezing limit. This leads to the prediction of peaks in the power spectrum of the relic Hawking radiation analogous to the Sakharov oscillations in the CMBR.

Many of the features discussed in this paper may be valid in the instance of non-spherical initial fields; i.e., non-zero initial perturbations. In this case, there will be cross terms in the classical action involving the initial and final field configurations. One can subsequently diagonalise the classical action resulting in a product of squeezed states, and so the results in this paper are valid. For rotating black holes, similar conclusions to those arrived at in this paper are possible. One must also include the fermionic perturbations, which are described by anti-commuting Grassmann variables.

## Acknowledgements

I thank Peter D'Eath and Ian Moss for helpful discussions, and the EPSRC for funding.

## Appendix A – Derivation of the wave function eqn.(7)

To arrive at eqn.(7), we set  $a = e^{-ik_j T}$ ,  $y = 0$  and  $x = x_j$  in Mehler's formula [9]

$$(1 - a^2)^{\frac{1}{2}} \sum_{p=0}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} \frac{a^p H_p(x) H_p(y)}{2^p p!} = e^{\frac{1}{2}(x^2-y^2)} e^{-\frac{(x-ay)^2}{1-a^2}}, \quad (66)$$

where

$$H_p(x) = (-1)^p e^{x^2} \frac{d^p}{dx^p} e^{-x^2} \quad (67)$$

are Hermite polynomials which satisfy

$$\frac{1}{2^p p! \pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx e^{-x^2} [H_p(x)]^2 = 1. \quad (68)$$

Then,

$$[2\pi i \sin(k_j T)]^{-\frac{1}{2}} e^{\frac{1}{2} i x_j^2 \cot(k_j T)} = \sum_{p=0}^{\infty} e^{-i E_p T} \psi_p(x_j) \psi_p(0), \quad (69)$$

where  $E_p = k_j(p + \frac{1}{2})$  and

$$\psi_p(x_j) = \frac{e^{-\frac{1}{2}x_j^2} H_p(x_j)}{(2^p p! \pi^{\frac{1}{2}})^{\frac{1}{2}}}, \quad (70)$$

i.e.

$$\int_{-\infty}^{\infty} dx_j |\psi_p(x_j)|^2 = 1.$$

Only even terms contribute to the sum in eqn.(69) as [8]

$$H_{2p}(0) = \frac{(-1)^p (2p)!}{p!}, \quad (71)$$

$$H_{2p+1}(0) = 0. \quad (72)$$

Hence,

$$[2\pi i \sin(k_j T)]^{-\frac{1}{2}} e^{\frac{1}{2} i x_j^2 \cot(k_j T)} = \sum_{p=0}^{\infty} e^{-i E_{2p} T} \psi_{2p}(x_j) \psi_{2p}(0). \quad (73)$$

In addition, set  $x_j = (\frac{k_j}{V})^{\frac{1}{2}} \text{Re}(A_j)$ ,  $y_j = (\frac{k_j}{V})^{\frac{1}{2}} \text{Im}(A_j)$ . Then,

$$\begin{aligned} \frac{e^{\frac{1}{2} i k_j V^{-1} |A_j|^2 \cot(k_j T)}}{2\pi i \sin(k_j T)} &= \sum_{p,p'=0}^{\infty} e^{-i(E_{2p} + E_{2p'})T} \psi_{2p}(x_j) \psi_{2p'}(y_j) \psi_{2p}(0) \psi_{2p'}(0) \\ &= \sum_{p=0}^{\infty} e^{-2i E_p T} \sum_{p'=0}^p \psi_{2p'}(x_j) \psi_{2p-2p'}(y_j) \psi_{2p'}(0) \psi_{2p-2p'}(0) \\ &= \frac{e^{-\frac{k_j}{2V} |A_j|^2}}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p e^{-2i E_p T}}{2^{2p} p!} \sum_{p'=0}^p \frac{p!}{p'!(p-p')!} H_{2p'}(x_j) H_{2p-2p'}(y_j), \end{aligned}$$

where  $V^{-1} = \Delta k_j$ . In the limit  $|T| \rightarrow 0_+$  (or rather  $k_j |T| \ll 1$ ), as the eigenfunctions  $\psi_p(x_j)$  form a complete orthonormal set

$$\begin{aligned} \lim_{k_j |T| \rightarrow 0_+} [2\pi i \sin(k_j T)]^{-1} e^{i \frac{k_j}{2V} |A_j|^2 \cot(k_j T)} &= \delta(x_j) \delta(y_j) \\ &\equiv \delta^{(2)}((\frac{k_j}{V})^{\frac{1}{2}} A_j). \end{aligned} \quad (74)$$

This suggests that  $A_j \rightarrow 0$  as  $k_j |T| \rightarrow 0_+$ , agreeing with our initial conditions of very weak perturbations. We now use the identity eqn.(13). Taking the product over all  $j$  and introducing a normalisation factor  $\hat{N}$ , then

$$\begin{aligned} \Psi[\{a_j\}; T] &= \prod_j \Psi_j(a_j; T) \\ &= \hat{N} e^{-i \frac{1}{2} M_I T} e^{-\frac{1}{2} \sum_j \frac{k_j}{V} |A_j|^2} \prod_j \sum_{p=0}^{\infty} e^{-2i E_p T} L_p \left( \frac{k_j}{V} |A_j|^2 \right), \end{aligned} \quad (75)$$

which is eqn.(7), where we have included the contribution from the time-like boundary. One can confirm that eqn.(75) gives us the first part of eqn.(7) by simply using the generating function for Laguerre polynomials [8].

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